Similarity Solutions of the Force-free Magnetic Field Equations

E. W. Richter

Institut für Mathematische Physik, Technische Universität Braunschweig, Mendelssohnstraße 3, D-38106 Braunschweig, Germany

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Force-free magnetic fields are described as solutions of special nonlinear partial differential equations which are replaced frequently through linear equations. To record the diversity of the structures of these fields, a discussion of the nonlinear equations is necessary. For this purpose the method of similarity analysis is used. The Lie symmetry groups admitted by the nonlinear equations for force-free magnetic fields are presented. To record and classify the different types of group-invariant solutions, one- and two-dimensional optimal systems of subalgebras are listed. The reduced equations of the two-dimensional optimal system are systems of ordinary differential equations, and their solutions define similarity solutions which are force-free magnetic fields. Only in one case is it necessary to calculate similarity solutions numerically. The corresponding reduced equations are a nonautonomous dynamical system with the similarity variable in the place of time.

Key words: Magnetic fields, Nonlinear PDE, Plasma equilibrium, Similarity solutions.

Introduction

Magnetic fields described by

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

are called force-free. They were first discussed by Lundquist [1] and later by other authors, e.g. Lüst and Schlüter [2]. The determination of field configurations for force-free magnetic fields and the development of methods to calculate them are usually not based on (1). Many authors use

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

as condition for force-free magnetic fields, where α is either a constant or a function of position and/or time (for references see, e.g., Priest [3]). Condition (2) with $\boldsymbol{v}$ instead of $\boldsymbol{B}$ is known in fluid mechanics as the condition for Beltrami fields. From $\nabla \cdot \boldsymbol{B} = 0$ follows

$$\boldsymbol{B} \cdot \nabla \alpha = 0. \tag{3}$$

Therefore, if α is a function of position, it has to be a constant along the magnetic lines of force.

If α is a given function for all solutions of (2), the problem is a linear one. But (1) describes a nonlinear problem since the sum of two different solutions of these equations does not, in general, produce a third solution. Of course, (2) is still valid in the nonlinear case and may be used to calculate the function α as

$$\alpha = \frac{1}{B^2} (\boldsymbol{B} \cdot \nabla \times \boldsymbol{B}). \tag{4}$$

In contrast to the linear case, this function α can be different for different solutions of (1).

The concept of force-free magnetic fields has been used in connection with several physical situations of plasma physics, for example in modeling the coronal magnetic fields of the Sun (e.g. Low and Lou [4], Browning [5]) or in the context with plasma relaxation theory and reversed field pinches of the magnetic confinement physics (cf. Taylor [6], Ortolani [7]). The above mentioned linear problem for force-free magnetic fields seems to be inappropriate for investigations of some interesting problems, especially of solar physics and astrophysics.

In this paper the tools of the similarity analysis will be used for determining exact solutions of the nonlinear equations (1). The similarity analysis for differential equations is well described in the literature (Ovsiannikov [8], Olver [9], Bluman and Kumei [10]). Therefore in what follows the details of the procedures are omitted.

In Sect. 1 the Lie point symmetry group admitted by (1) is given. In Sect. 2 the classification of possible similarity solutions is carried out and optimal systems are determined. In Sect. 3 some similarity solutions of (1) are presented.

Reprint requests to Prof. E. W. Richter, Institut für Mathematische Physik, Technische Universität Braunschweig, Mendelssohnstraße 3, D-38106 Braunschweig.

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1. Lie Symmetries of the Equations for Force-free Magnetic Fields

Transformations acting on the independent and dependent variables of a system of differential equations with the property that solutions are transformed to other solutions of this system are called symmetry transformations. The symmetry group of a system of differential equations is the largest local group of symmetry transformations admitted by the system. As usually, only connected local Lie groups of symmetries are considered in this paper. The infinitesimal generators of a Lie symmetry group are vector fields which are elements of the Lie algebra corresponding to the Lie group. Since every element of a Lie symmetry group can be obtained by exponential maps for suitable elements of the corresponding Lie algebra, the explicit determination of the symmetry group of a system of differential equations is reduced to the calculation of infinitesimal generators providing a basis for the admitted Lie algebra. The Lie algebra of infinitesimal symmetries of the Eqs. (1) is spanned by the eight vector fields (Richter [11]):

$$v_1 = \frac{\partial}{\partial x}, \quad v_2 = \frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial z},$$

$$v_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + B_y\frac{\partial}{\partial B_x} - B_x\frac{\partial}{\partial B_y},$$

$$v_5 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} + B_x\frac{\partial}{\partial B_z} - B_z\frac{\partial}{\partial B_x},$$

$$v_6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + B_z\frac{\partial}{\partial B_y} - B_y\frac{\partial}{\partial B_z},$$

$$v_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

$$v_8 = B_x\frac{\partial}{\partial B_x} + B_y\frac{\partial}{\partial B_y} + B_z\frac{\partial}{\partial B_z}. \tag{5}$$

The commutator of the Lie algebra, defined through $[v_i, v_j] = v_i(v_j) - v_j(v_i)$, yields the commutator table (Table 1).

The base vectors v_i, $i = 1, \ldots, 8$, generate one-parameter groups G_i, $i = 1, \ldots, 8$. The symmetry groups G_1, G_2, G_3 are space translations and G_4, G_5, G_6 are space rotations. The group G_7 represents a space scaling symmetry while G_8 demonstrates the magnetic field scaling of the system (1). Thus the list of symmetries admitted by this system is rather small. The system (2) with $\alpha \in \mathbb{R}$ admits a 7-parameter Lie group

Table 1. Commutator table of the Lie algebra with base vectors (5).

$[v_\downarrow, v_\rightarrow]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	0	0	0	$-v_2$	v_3	0	v_1	0
v_2	0	0	0	v_1	0	$-v_3$	v_2	0
v_3	0	0	0	0	$-v_1$	v_2	v_3	0
v_4	v_2	$-v_1$	0	0	$-v_6$	v_5	0	0
v_5	$-v_3$	0	v_1	v_6	0	$-v_4$	0	0
v_6	0	v_3	$-v_2$	$-v_5$	v_4	0	0	0
v_7	$-v_1$	$-v_2$	$-v_3$	0	0	0	0	0
v_8	0	0	0	0	0	0	0	0

where the corresponding Lie algebra is formed by $v_1, \ldots, v_6$, and v_8 of (5). As is well-known, linear partial differential equations always admit a trivial infinite-parameter Lie group of transformations in addition to finite-parameter Lie groups. In case of (2) these trivial transformations are $(\boldsymbol{x}, \boldsymbol{B}) \to (\boldsymbol{x}, \boldsymbol{B} + \varepsilon \boldsymbol{B}_0)$, where $\boldsymbol{B}_0(\boldsymbol{x})$ is any function satisfying (2).

In this paper only the system (1) is investigated. The full symmetry eight-parameter group G of (1) can be generated by the one-parameter transformation groups $G_1, \ldots, G_8$. In particular, an arbitrary group transformation $g \in G$ can be represented uniquely in the form

$$g = \exp(\varepsilon_8 v_8) \cdot \ldots \cdot \exp(\varepsilon_1 v_1) \tag{6}$$

for suitable $(\varepsilon_1, \ldots, \varepsilon_8) \in \mathbb{R}^8$. This element generates the transformation

$$(\boldsymbol{x}, \boldsymbol{B}) \to (\boldsymbol{a} + e^{\varepsilon_7} R\, \boldsymbol{x},\; e^{\varepsilon_8} R\, \boldsymbol{B}) \tag{7}$$

with $\boldsymbol{a} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and the 3×3 orthogonal matrix $R = R_4 R_5 R_6$, where

$$R_4 = \begin{pmatrix} \cos\varepsilon_4 & \sin\varepsilon_4 & 0 \\ -\sin\varepsilon_4 & \cos\varepsilon_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_5 = \begin{pmatrix} \cos\varepsilon_5 & 0 & -\sin\varepsilon_5 \\ 0 & 1 & 0 \\ \sin\varepsilon_5 & 0 & \cos\varepsilon_5 \end{pmatrix},$$

$$R_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_6 & \sin\varepsilon_6 \\ 0 & -\sin\varepsilon_6 & \cos\varepsilon_6 \end{pmatrix}. \tag{8}$$

The corresponding action on solutions of (1) shows that if $\boldsymbol{B} = \boldsymbol{f}(\boldsymbol{x})$ is a solution, so are

$$\boldsymbol{B} = e^{\varepsilon_8} R \boldsymbol{f}(e^{-\varepsilon_7} R^{-1}(\boldsymbol{x} - \boldsymbol{a})), \quad (\varepsilon_1, \ldots, \varepsilon_8) \in \mathbb{R}^8, \tag{9}$$

where R^{-1} is the inverse of matrix R.

2. Optimal Systems of Subgroups of the Full Symmetry Group G

Every solution of system (1) which is itself invariant under some subgroup of the symmetry group G is called group-invariant or a similarity solution. Let $H \subset G$ be an s-parameter subgroup of the 8-parameter symmetry group G. Every H-invariant solution, i.e. every solution of (1) which is invariant for all transformations $h \in H$, can be obtained by solving a system of partial differential equations involving $s \leq 3$ fewer independent variables then the Equations (1), called reduced system. Therefore, if $s = 2$, one gets a system of ordinary differential equations, and for $s = 3$ the reduced system would be algebraic. With every solution of a reduced system one finds similarity solutions of system (1) by substituting back the used transformations. In many cases it is possible to get in this way similarity solutions in analytic forms. Otherwise one can solve the reduced systems of ordinary differential equations numerically.

For classifying group-invariant solutions one needs a criterion of equivalence under which all possible subgroups of G are separated into nonintersecting classes of subgroups. For any $g \in G$ with $g \notin H$, an H-invariant solution is transferred to a gHg^{-1}-invariant solution. The two subgroups H and gHg^{-1} are called conjugate. An optimal system Θ_s of G is a list of s-parameter subgroups of G which contains one representative of each conjugacy class. Therefore every s-parameter subgroup of G is conjugate to precisely one member in the optimal system Θ_s. If one has one similarity solution for all subgroups of this list, every other similarity solution which is invariant under some s-parameter subgroup of G can be found by a suitable element $g \in G$.

The correspondence between Lie subgroups and Lie subalgebras leads to an analog formulation for optimal systems of subalgebras. If one knows an optimal system of subalgebras, a corresponding optimal system of subgroups can be constructed by application of the exponential map to every subalgebra. Therefore, techniques for determining optimal subalgebraic systems are very important (for references and a summary of the techniques see, e.g., Kötz [12]).

The results of classifications for one- and two-dimensional optimal systems of subalgebras of the Lie algebra given by (5) are placed in Tables 2 and 3. The vectorfields enclosed by brackets generate the different subalgebras $\mathscr{H}$.

Table 2. The one-dimensional optimal system Θ_1 where $a \geq 0$ and $b \in \mathbb{R}$.

$\mathscr{H}(v_1+v_6+b\,v_8)$	$\mathscr{H}(-v_1+v_6+b\,v_8)$	$\mathscr{H}(a\,v_6+v_7+b\,v_8)$
$\mathscr{H}(v_1+b\,v_8)$	$\mathscr{H}(v_6+b\,v_8)$	

Table 3. The two-dimensional optimal system Θ_2 where $a \geq 0$ and $b \in \mathbb{R}$.

$\mathscr{H}(v_1, v_2)$	$\mathscr{H}(v_1, v_2+v_8)$	$\mathscr{H}(v_1+v_8, v_2+a\,v_8)$
$\mathscr{H}(v_3, v_4+a\,v_8)$	$\mathscr{H}(v_3+v_8, v_4+a\,v_8)$	$\mathscr{H}(v_3-v_8, v_4+a\,v_8)$
$\mathscr{H}(v_4+a\,v_8, v_7+b\,v_8)$		

3. Similarity Solutions of Equations (1)

Subgroups of the two-dimensional optimal system Θ_2 (see Table 3) are considered. In the following the results are only assembled, and trivial solutions $\boldsymbol{B} = \text{const}$ are not mentioned. Note that every solution which is given in the following can be transformed according to (9) in order to obtain new and mostly more complicated similarity solutions.

A) $\mathscr{H}^1(v_1, v_2)$

B_x, B_y, B_z are functions which cannot depend on the variables x and y. The reduced equations have nontrivial solutions only for $B_z = 0$, and in this case they merely require $B_x^2(z) + B_y^2(z) = B_0^2 \equiv \text{const}$, i.e., a constant energy density of the magnetic field.

Equation (4) yields

$$\alpha = \frac{1}{B_0^2}\left(B_y \frac{\mathrm{d}B_x}{\mathrm{d}z} - B_x \frac{\mathrm{d}B_y}{\mathrm{d}z}\right). \tag{10}$$

An admitted similarity solution is

$$B_x = c_1 \cos(u) + c_2 \sin(u),$$
$$B_y = \pm(c_2 \cos(u) - c_1 \sin(u)), \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$ with $c_1^2 + c_2^2 = B_0^2$ and an arbitrary differentiable function $u(z)$. From (10) one gets $\alpha = \pm\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)$. Therefore (11) describes solutions of the linear problem (2) only if the functions $u(z)$ have identical derivatives.

B) $\mathscr{H}^2(v_1, v_2+v_8)$

B_x, B_y, B_z are functions merely of the two independent variables y and z. The reduced equations are ordinary differential equations with z as independent variable. They have nontrivial solutions only for $B_z \neq 0$, and in this case one finds the similarity solutions

$$B_x = \gamma \exp(y)\,(c_1 \cos(\varkappa z) + c_2 \sin(\varkappa z)),$$
$$B_y = \varkappa \exp(y)\,(c_1 \sin(\varkappa z) - c_2 \cos(\varkappa z)),$$
$$B_z = \exp(y)\,(c_1 \cos(\varkappa z) + c_2 \sin(\varkappa z)) \tag{12}$$

with $c_1, c_2, \gamma \in \mathbb{R}$ and $\varkappa = \sqrt{1+\gamma^2}$. Finally it follows $\alpha = -\gamma = \text{const.}$

C) $\mathscr{H}^3(v_1 + v_8, v_2 + a v_8)$, where $a \geq 0$

The reduced equations are ordinary differential equations with z as independent variable. The solutions of these equations produce the similarity solutions

$$B_x = \frac{\exp(v)}{1+a^2} \cdot ((\varkappa c_1 - \gamma a c_2)\sin(\varkappa z) - (\varkappa c_2 + \gamma a c_1)\cos(\varkappa z)),$$
$$B_y = \frac{\exp(v)}{1+a^2} \cdot ((\varkappa a c_1 + \gamma c_2)\sin(\varkappa z) + (\gamma c_1 - \varkappa a c_2)\cos(\varkappa z)),$$
$$B_z = \exp(v)\,(c_2 \sin(\varkappa z) + c_1 \cos(\varkappa z)), \tag{13}$$

with $c_1, c_2, \gamma \in \mathbb{R}$ and $\varkappa = \sqrt{1+a^2+\gamma^2}$, $v = x + a y$. From (10) it follows $\alpha = \gamma = \text{const.}$

D) $\mathscr{H}^4(v_3, v_4 + a v_8)$, where $a \geq 0$

Convenient formulations are obtained by introducing the cylindrical coordinates ϱ, φ, z. One can show that $B_\varrho, B_\varphi, B_z$ are functions merely of the two independent variables ϱ and φ. If $\boldsymbol{B}(\varrho, \varphi + 2\pi, z) = \boldsymbol{B}(\varrho, \varphi, z)$ is required, only the case $a = 0$ is possible, and $B_\varrho, B_\varphi, B_z$ are functions of the sole variable ϱ.

For $a = 0$ and $B_\varrho \neq 0$ one gets similarity solutions with $\alpha = 0$, i.e. $\boldsymbol{B}$ describes a potential field.

If $B_\varrho = 0$ is assumed, only one reduced equation remains:

$$B_\varphi \frac{d(\varrho B_\varphi)}{d\varrho} + \varrho B_z \frac{dB_z}{d\varrho} = 0, \tag{14}$$

and (4) reduces to

$$\alpha = \frac{1}{\varrho B^2}\left(B_z \frac{d(\varrho B_\varphi)}{d\varrho} - \varrho B_\varphi \frac{dB_z}{d\varrho}\right). \tag{15}$$

Substitution of (14) into (15) leads to

$$\alpha B_\varphi = -\frac{dB_z}{d\varrho}, \tag{16}$$

and that implies for (14) the form

$$\varrho \frac{d^2 B_z}{d\varrho^2} + \left(1 - \frac{\varrho}{\alpha}\frac{d\alpha}{d\varrho}\right)\frac{dB_z}{d\varrho} + \varrho \alpha^2 B_z = 0. \tag{17}$$

Many of the force-free magnetic fields which have been discussed in the literature are solutions of (14) with (16). Note that all these fields are similarity solutions of (1) or (2).

For the case $\alpha \in \mathbb{R}$, (17) is the well known Bessel equation with respect to the independent variable $\lambda = \alpha \varrho$. The solution of this equation, which is regular at $\lambda = 0$ is $B_z = B_0 J_0(\lambda)$, whereby J_0 is the Bessel function of order 0 and B_0 is a constant fixing the field strength. According to (16) one has $B_\varphi = B_0 J_1(\lambda)$, where J_1 is the Bessel function of order 1. This magnetic field describes the so called Lundquist field (Lundquist [1]).

Other assumptions are possible for the function $\alpha(\varrho)$ in (17). For example a simple analytical solution of this equation can be calculated with $\alpha = \frac{\varkappa}{\varrho}$ and $0 < |\varkappa| < 0.5$:

$$B_z = c_1 \varrho^{-1/2(1-\gamma)} + c_2 \varrho^{-1/2(1+\gamma)},$$
$$B_\varphi = \frac{1}{2\varkappa}((1-\gamma) c_1 \varrho^{-1/2(1-\gamma)} + (1+\gamma) c_2 \varrho^{-1/2(1+\gamma)}), \tag{18}$$

where $c_1, c_2 \in \mathbb{R}$ and $\gamma = \sqrt{1 - 4\varkappa^2}$.

For $|\varkappa| = 0.5$ follows

$$B_z = \varrho^{-1/2}(c_1 + c_2 \ln \varrho),$$
$$B_\varphi = \operatorname{sgn}(\varkappa)\,\varrho^{-1/2}(c_1 + c_2(\ln \varrho - 2)). \tag{19}$$

Force-free magnetic fields of these and more general forms have been discussed by Chiuderi et al. [13].

If $\alpha(B_z(\varrho))$ is assumed, (17) may be written

$$\varrho \frac{d^2 B_z}{d\varrho^2} + \frac{dB_z}{d\varrho} - \frac{\varrho}{\alpha}\frac{d\alpha}{dB_z}\left(\frac{dB_z}{d\varrho}\right)^2 + \varrho \alpha^2 B_z = 0. \tag{20}$$

This equation has been discussed in Cartesian coordinates by Emtes and Kovbasenko [14].

E) $\mathscr{H}^5(v_3 + b v_8, v_4 + a v_8)$, where $a \geq 0$ and $b = \pm 1$

Similar like at $\mathscr{H}^4$, it is convenient to introduce the cylindrical coordinates ϱ, φ, z and to require

$\boldsymbol{B}(\varrho, \varphi+2\pi, z)=\boldsymbol{B}(\varrho, \varphi, z)$. Hence only the case $a=0$ is considered.

If $B_\varrho=0$ is assumed, one gets only a zero magnetic field.

If one chooses $B_\varrho \neq 0$, then $\alpha=\text{const}$ follows and one finds the similarity solutions

$$B_\varrho = c\, B_\varphi,$$
$$B_\varphi = -\frac{\exp(b z)}{\sqrt{1+c^2}}\,(c_1 J_1(\varkappa \varrho)+c_2 Y_1(\varkappa \varrho)),$$
$$B_z=\exp(b z)\,(c_1 J_0(\varkappa \varrho)+c_2 Y_0(\varkappa \varrho)) \tag{21}$$

with $c, c_1, c_2 \in \mathbb{R}$, $c \neq 0$, $\varkappa=\frac{b}{c}\sqrt{1+c^2}$ and $\alpha=-\frac{b}{c}$. Y_0 and Y_1 are Neumann functions of order 0 and 1.

F) $\mathscr{H}^6(v_4+a v_8,\, v_7+b v_8)$, where $a \geq 0$ and $b \in \mathbb{R}$

Similar like at $\mathscr{H}^4$, it is again convenient to introduce the cylindrical coordinates ϱ, φ, z and to require $\boldsymbol{B}(\varrho, \varphi+2\pi, z)=\boldsymbol{B}(\varrho, \varphi, z)$. This leads again to the case $a=0$, and it follows

$$B_\varrho=z^b \zeta_1(\lambda), \quad B_\varphi=z^b \zeta_2(\lambda), \quad B_z=z^b \zeta_3(\lambda) \tag{22}$$

with the similarity variable $\lambda=\frac{\varrho}{z}$ which may be interpreted as $\tan\vartheta$, where ϑ is the polar angle which varies from 0 to π.

If $B_\varrho=0$ and $b \neq 0$ are assumed, one finds

$$B_\varphi=c\,\varrho^b,\; B_z=\pm c\sqrt{-\left(1+\frac{1}{b}\right)}\,\varrho^b \tag{23}$$

with $c \in \mathbb{R}$ and

$$\alpha=\pm\frac{\sqrt{-b(1+b)}}{\varrho}. \tag{24}$$

Therefore real-valued solutions are only possible for $-1 \leq b<0$. Note that (23) and (24) follow without additional assumptions in contrast to (18), where an assumption for α was required within (17).

If $B_\varrho \neq 0$ is admitted, one gets the reduced system

$$\begin{aligned}
&\lambda\zeta_3(\lambda\zeta_1'+\zeta_3')+\lambda\zeta_2\zeta_2'-b\,\lambda\zeta_1\zeta_3+\zeta_2^2=0,\\
&\lambda(\zeta_1-\lambda\zeta_3)\,\zeta_2'+b\,\lambda\zeta_2\zeta_3+\zeta_1\zeta_2=0,\\
&\zeta_1(\lambda\zeta_1'+\zeta_3')+\lambda\zeta_2\zeta_2'-b(\zeta_1^2+\zeta_2^2)=0,\\
&\lambda(\zeta_1'-\lambda\zeta_3')+b\,\lambda\zeta_3+\zeta_1=0,
\end{aligned} \tag{25}$$

where $b \in \mathbb{R}$ is a constant parameter and the prime $'$ denotes derivation with respect to λ. This system can be written as a nonautonomous system of evolution equations:

$$\begin{aligned}
\zeta_1'&=\frac{1}{1+\lambda^2}\left(b(\lambda\zeta_1-\zeta_3)-\frac{\zeta_1}{\lambda}+\frac{(1+b)\lambda\zeta_2^2}{\zeta_1-\lambda\zeta_3}\right),\\
\zeta_2'&=-\frac{\zeta_2}{\zeta_1-\lambda\zeta_3}\left(b\zeta_3+\frac{\zeta_1}{\lambda}\right),\\
\zeta_3'&=\frac{1}{1+\lambda^2}\left(\zeta_1+b(\zeta_1+\lambda\zeta_3)+\frac{(1+b)\zeta_2^2}{\zeta_1-\lambda\zeta_3}\right).
\end{aligned} \tag{26}$$

The system (26) may be discussed as a nonautonomous dynamical system with the independent variable λ in the place of time. Note that this system admits the symmetry transformation $(\lambda, \zeta_1, \zeta_2, \zeta_3) \to(-\lambda, \zeta_1, \zeta_2,-\zeta_3)$. Therefore it is sufficient to discuss (26) for $\lambda \geq 0$. Real-valued solutions of (25) or (26) which are independent of λ are equilibrium points or fixed points of the system. From (25) one finds in the case $b=0$ the fixed points

$$\bar{\zeta}_1=\bar{\zeta}_2=0, \quad \bar{\zeta}_3 \in \mathbb{R}, \tag{27}$$

and in the case $b \neq 0$ the fixed point

$$\bar{\zeta}_1=\bar{\zeta}_2=\bar{\zeta}_3=0. \tag{28}$$

The energy density of the magnetic field is given by

$$\frac{1}{2\mu_0}B^2=\frac{z^{2b}}{\mu_0}u(\lambda) \tag{29}$$

with

$$u(\lambda)=\zeta_1^2+\zeta_2^2+\zeta_3^2. \tag{30}$$

Substituting (22) into (4), one obtains

$$\alpha=\frac{a(\lambda)}{z} \tag{31}$$

with

$$a(\lambda)=\frac{1}{u(\lambda)}\left(\lambda(\zeta_1\zeta_2'-\zeta_2\zeta_1')+\zeta_3\zeta_2'-\zeta_2\zeta_3'+\frac{1}{\lambda}\zeta_2\zeta_3\right). \tag{32}$$

From system (25) it follows by a simple calculation

$$u'(\lambda)=-\frac{2}{\lambda(1+\lambda^2)}\left(\zeta_1^2+\zeta_2^2-\lambda\zeta_1\zeta_3-b\lambda^2 u\right). \tag{33}$$

For physical reasons it may be important to consider magnetic fields with energy densities which are not increasing with λ. Therefore, taking (33) into account, in the following only nonpositive values of b are considered.

There is little physical interest in the case $u=\text{const}$ but it admits an analytical solution which is very helpful to obtain a general view of the types of numerical solutions of the system (25). Moreover this closed-form solution opens up the opportunity to test the reliability and accuracy of the used numerical procedures. For $u'=0$ one obtains from (33) in the case of $b=0$

$$\zeta_1^2+\zeta_2^2=\lambda\,\zeta_1\,\zeta_3. \tag{34}$$

With $u=\text{const}$, the third equation of the system (25) yields $(\zeta_1-\lambda\,\zeta_3)\,\zeta_3'=0$, i.e. $\zeta_3=\text{const}$. Therefore $\zeta_1^2+\zeta_2^2=\text{const}$ is valid, and in accordance with the remaining equations of (25) it follows from (34)

$$\zeta_1(\lambda)=\frac{C}{\lambda},\quad \zeta_2(\lambda)=\pm\zeta_1(\lambda)\sqrt{K\,\lambda^2-1},\quad \zeta_3(\lambda)=CK, \tag{35}$$

where $C, K\in\mathbb{R}$ with $K\geq\lambda_{\text{in}}^{-2}$ are given by initial conditions. Thus from (22) with $b=0$ follows immediately that (35) is a similarity solution of the system (1). Moreover, the existence of a solution of the dynamical system (26) with $u=\text{const}$ points out that the energy density of the magnetic field presumably also in the case $b\neq 0$ cannot used as a Lyapunov function in the neighborhood of the fixed point (28).

Initial conditions for the point $\lambda_{\text{in}}=0.1$, which are suggested by (35), may be

$$\zeta_{1\,\text{in}}=\zeta_{2\,\text{in}}=0.1,\quad \zeta_{3\,\text{in}}=2, \tag{36}$$

or

$$\zeta_{1\,\text{in}}=0.1,\quad \zeta_{2\,\text{in}}=0.2,\quad \zeta_{3\,\text{in}}=5, \tag{37}$$

and so on.

If $b<0$ is considered in (26), one finds in the particular case $b=-1$ that the system gets the reduced form

$$\zeta_1'=\frac{1}{1+\lambda^2}\,\zeta_3-\frac{1}{\lambda}\,\zeta_1,\quad \zeta_2'=-\frac{1}{\lambda}\,\zeta_2,\qquad \zeta_3'=-\frac{\lambda}{1+\lambda^2}\,\zeta_3. \tag{38}$$

The general solution of this system substituted in (22) yields

$$B_\varrho=\frac{1}{\varrho}\left(c_1-\frac{c_3}{\sqrt{1+\lambda^2}}\right),\quad B_\varphi=\frac{c_2}{\varrho},\quad B_z=\frac{c_3}{z\sqrt{1+\lambda^2}} \tag{39}$$

with $c_1, c_2, c_3\in\mathbb{R}$. Substituting (38) into (32) one finds $\alpha=0$. Therefore the corresponding similarity solution describes a potential field.

In case of $b<0$ and $b\neq -1$ numerical solutions to the system (26) for real-valued functions ζ_i, $i=1, 2, 3$, have been carried out on some specified λ-intervals $[\lambda_{\min}, \lambda_{\max}]$ with suitably chosen initial conditions $\zeta_{j\text{in}}=\zeta_j(\lambda_{\text{in}})$, $j=1, 2, 3$, where $\lambda_{\text{in}}\equiv\lambda_{\min}$ is chosen. The numerical calculations were done with the aid of the computer programs of Mathematica ([15]).

The system (26) is singular at $\lambda=0$ and, if $\zeta_2(\lambda)\neq 0$, at $\zeta_1(\lambda)=\lambda\,\zeta_3(\lambda)$. Two cases arise for the condition $\zeta_{2\,\text{in}}=0$. If $\zeta_{1\,\text{in}}=\lambda_{\text{in}}\,\zeta_{3\,\text{in}}$ holds, it follows from (26) that $\zeta_{2\,\text{in}}'$ is not defined, hence an integration of the system is impossible. If $\zeta_{1\,\text{in}}\neq\lambda_{\text{in}}\,\zeta_{3\,\text{in}}$ is given, $\zeta_2(\lambda)=0$ holds always for $\lambda\in[\lambda_{\min}, \lambda_{\max}]$. Therefore in the following we assume that not only $\lambda_{\min}\neq 0$ but also $\zeta_{2\,\text{in}}\neq 0$ and $\zeta_{1\,\text{in}}\neq\lambda_{\text{in}}\,\zeta_{3\,\text{in}}$ are valid.

With suitably chosen initial conditions the numerical calculations have been carried out over the domain $0<\lambda_{\text{in}}\leq\lambda\leq\lambda_{\max}$. Critical λ-points of the system (26) are indicated with the aid of the denominator

$$\delta(\lambda):=\lambda\,\zeta_3(\lambda)-\zeta_1(\lambda) \tag{40}$$

by $\delta(\lambda_{\text{crit}})=0$. Numerical solutions of (26) may have singularities at critical λ-points. In this case one has $\lambda_{\max}=\lambda_{\text{crit}}$. But if $\zeta_2(\lambda_{\text{crit}})=0$ holds, solutions without singularities are possible. In what follows only solutions without singularities are considered. These solutions for $b<0$ are attracted by the fixed point (28) and the function $u(\lambda)$ of the energy density (29) decreases with increasing λ, frequently monotonously. Hence the end point $\lambda_{\max}$ of the λ-interval may be chosen in such a manner that the solution approaches the fixed point sufficiently. In connection with this, it is convenient to change the independent variable by

$$\eta=k\,\lambda \tag{41}$$

with $k>0$. Thus λ-intervals can be transformed into η-intervals with the unchanged end point $\eta_{\max}=1$, and with this condition it follows $k=\lambda_{\max}^{-1}$ and $\eta_{\text{in}}=k\,\lambda_{\text{in}}$. After a corresponding transformation of the system (26) one can carry out the calculation of the above mentioned solutions over the domain $0<\eta_{\text{in}}\leq\eta\leq 1$. In what follows, all so far introduced functions of the independent variable λ are thought redefined as functions of η.

Examples for admissible initial conditions in the case $b=0$ are given in (36) or (37), now for $\eta_{\text{in}}=k\,\lambda_{\text{in}}=0.1\,k$. In order to obtain a general view of solutions of the system (26) it looks reasonable to chose these or similar initial conditions for values $b<0$. The numerical studies have shown that the typical behavior of the solutions of the system (26) is already discovered if only one set of the above men-

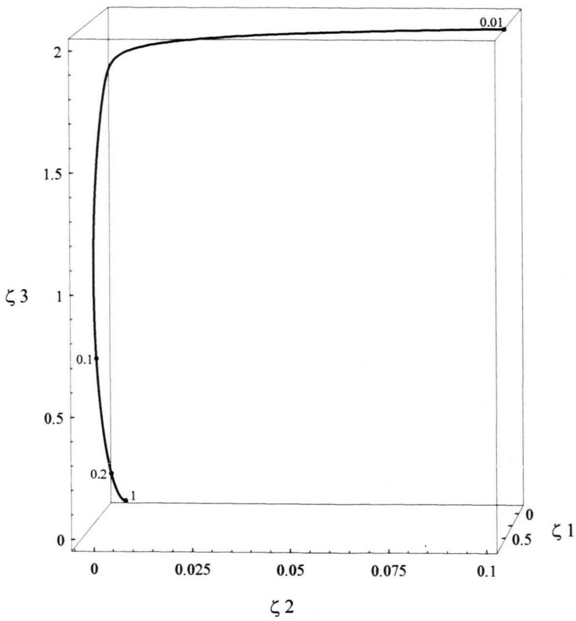

Fig. 1. Parametric plot of a solution $\{(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta)\}$ for (26) with $b=-2$ and (41) with $k=0.1$. Initial conditions are (36) with $\eta_{\text{in}}=0.01$. Some values of the η-parameter are indicated by dots. $\eta=1$ marks the point $\{0, 0, 0\}$.

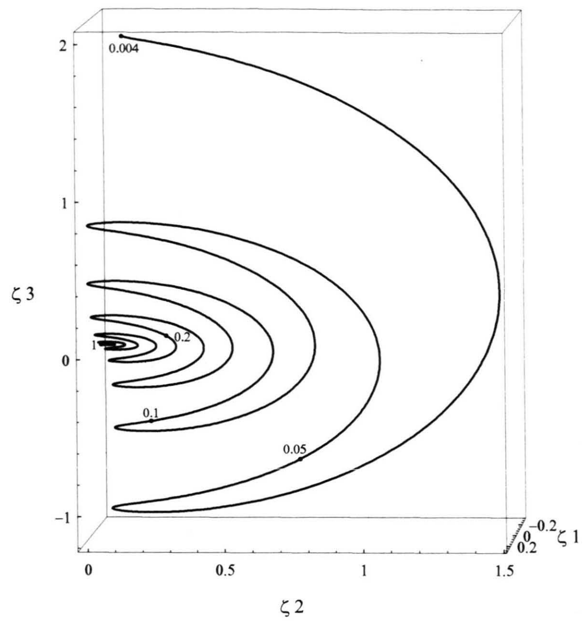

Fig. 3. Parametric plot of a solution $\{(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta)\}$ for (26) with $b=-3$ and (41) with $k=0.2$. Initial conditions are (36) with $\eta_{\text{in}}=0.004$. Some values of the η-parameter are indicated by dots. $\eta=1$ marks the point $\{0, 0, 0\}$.

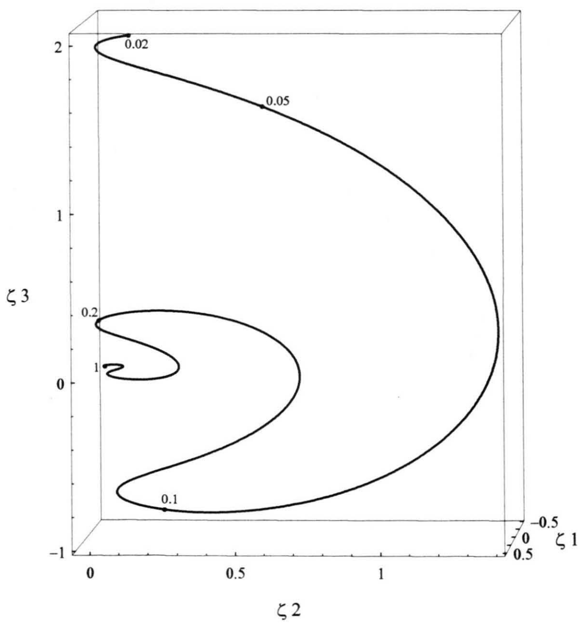

Fig. 2. Parametric plot of a solution $\{(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta)\}$ for (26) with $b=-3$ and (41) with $k=0.2$. Initial conditions are (36) with $\eta_{\text{in}}=0.02$. Some values of the η-parameter are indicated by dots. $\eta=1$ marks the point $\{0, 0, 0\}$.

tioned initial conditions is used. Therefore in the following Figures only some results of the calculations for the initial conditions (36) but for a wide range of the parameter $b<0$ are presented. More precisely, it is sufficient to represent solutions of (26) for values $b \leq -2$ because in the range $-2<b<0$ the numerical results are similar to those plotted in the case $b=-2$. Only the sign of the function $a(\eta)$ is reversed in the range $-1<b<0$, and this is easy to understand if one remembers that we found $\alpha=0$ in the case of $b=-1$.

In Figs. 1, 2, 3, and 9 are shown solutions of the system (26) which are essentially different. For these solutions the variations of energy density with η are similar to the variation which is presented for one case in Figure 6. Typical variations of the function α are shown in Figs. 7 and 8. From Fig. 12 one can learn that the functions $\zeta_2$and δ may have the same zeros. Therefore, singularities of the solutions are by no means necessary in the case $\delta(\eta)=0$.

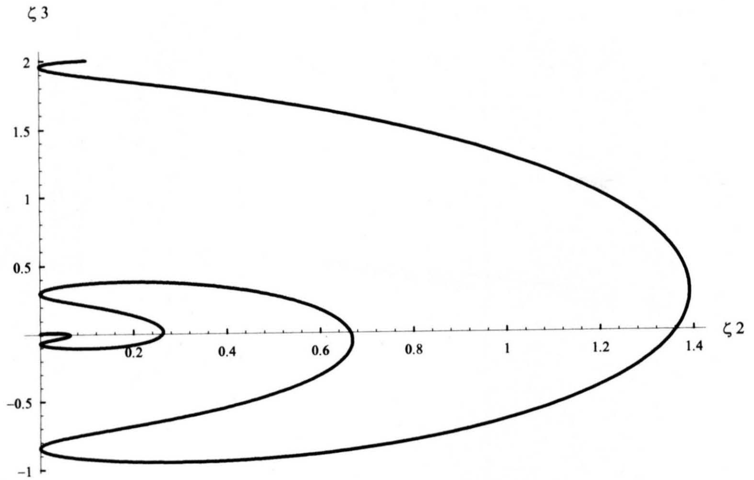

Fig. 4. Plot of the projection of the trajectory in Fig. 2 onto the (ζ_2, ζ_3)-plane. Initial point is $\{0.1, 2\}$.

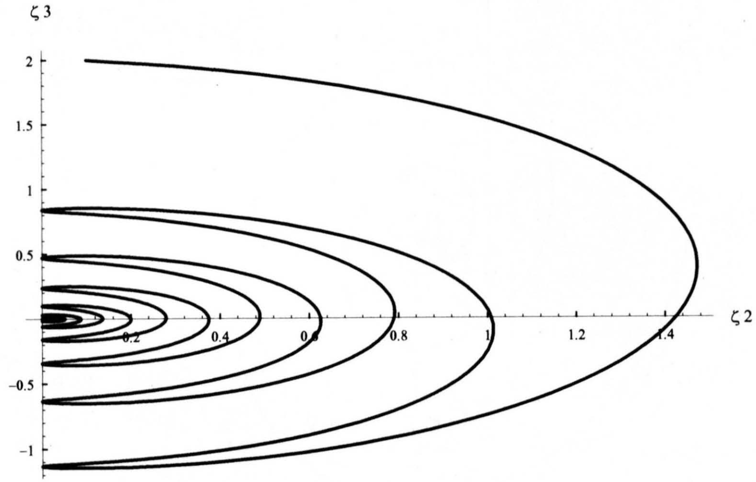

Fig. 5. Plot of the projection of the trajectory in Fig. 3 onto the (ζ_2, ζ_3)-plane. Initial point is $\{0.1, 2\}$.

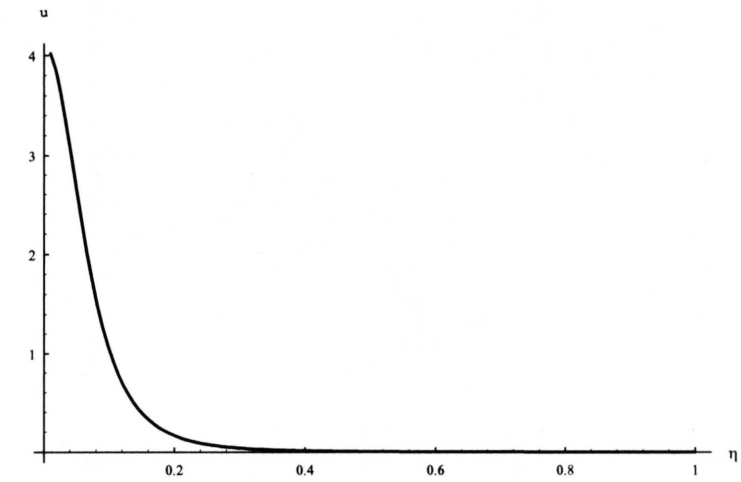

Fig. 6. Plot of $u(\eta)$ corresponding to (30) with the same conditions as in Figure 1. This function determines the η-development of the energy density (29).

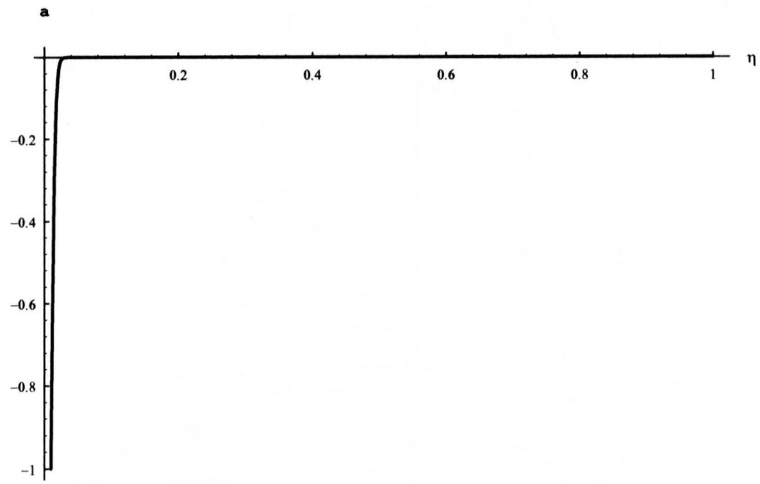

Fig. 7. Plot of $a(\eta)$ corresponding to (32) with the same conditions as in Figure 1. This function determines the η-development of α in (31).

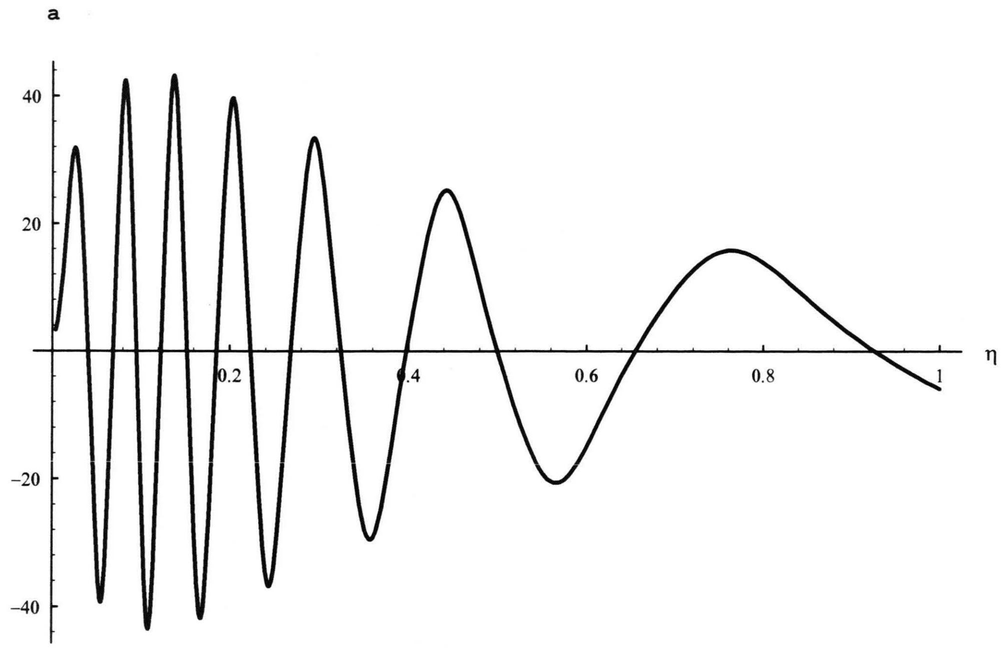

Fig. 8. Plot of $a(\eta)$ corresponding to (32) with the same conditions as in Figure 3. This function determines the η-development of α in (31).

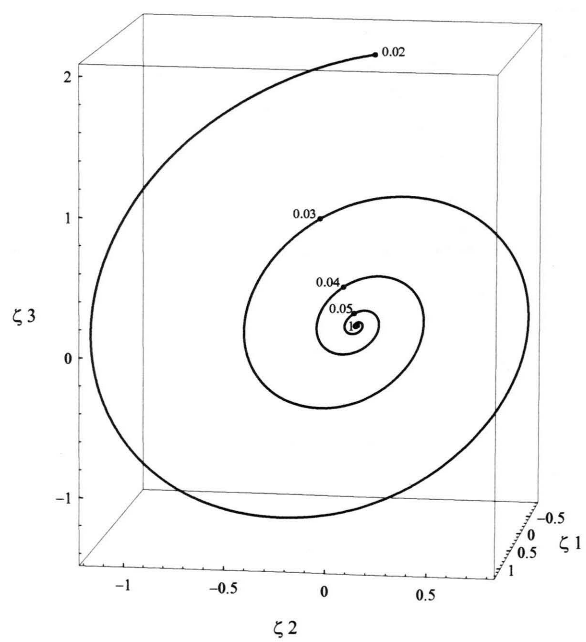

Fig. 9. Parametric plot of a solution $\{(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta)\}$ for (26) with $b=-99$ and (41) with $k=0.2$. Initial conditions are (36) with $\eta_{in}=0.02$. Some values of the η-parameter are indicated by dots. $\eta=1$ marks the point $\{0, 0, 0\}$.

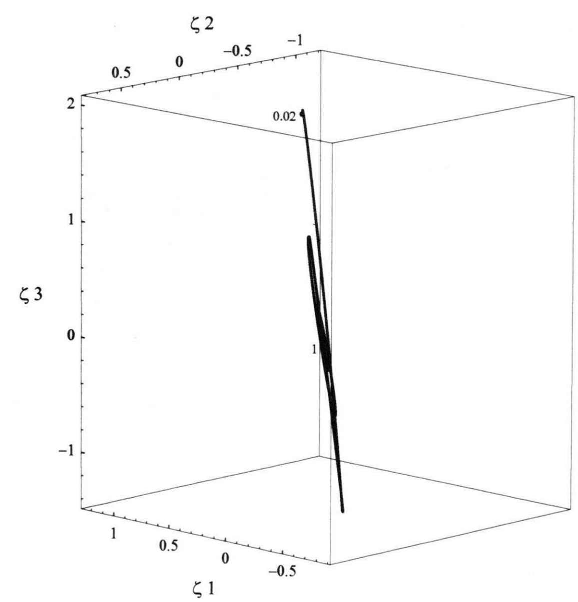

Fig. 10. Parametric plot of $\{(\zeta_1(\eta), \zeta_2(\eta), \zeta_3(\eta)\}$ as in Fig. 9, but the view point is shifted in order to show that the trajectory moves approximately within a plane.

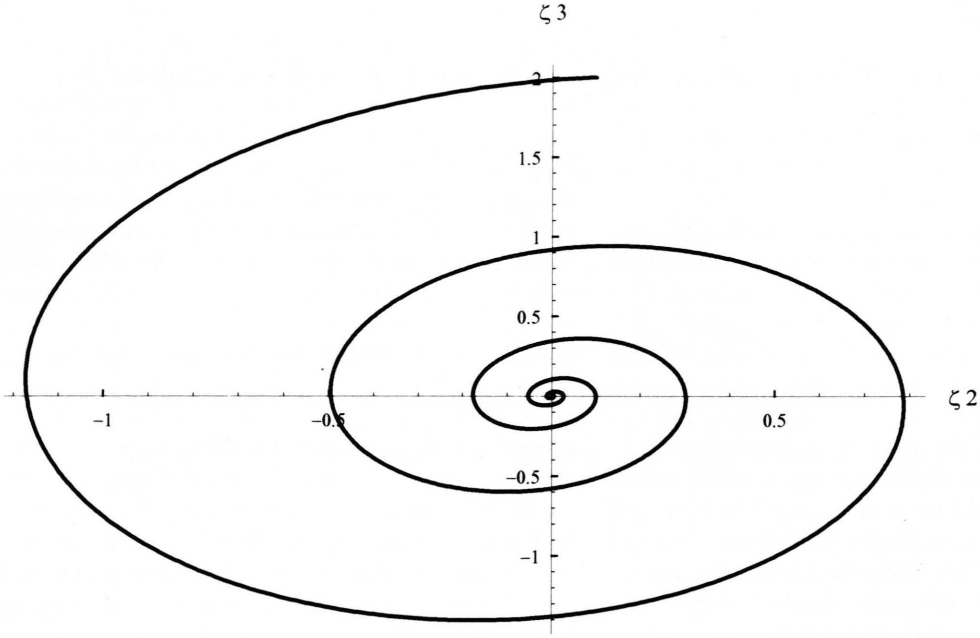

Fig. 11. Plot of the projection of the trajectory in Fig. 9 onto the (ζ_2, ζ_3)-plane. Initial point is $\{0.1, 2\}$.

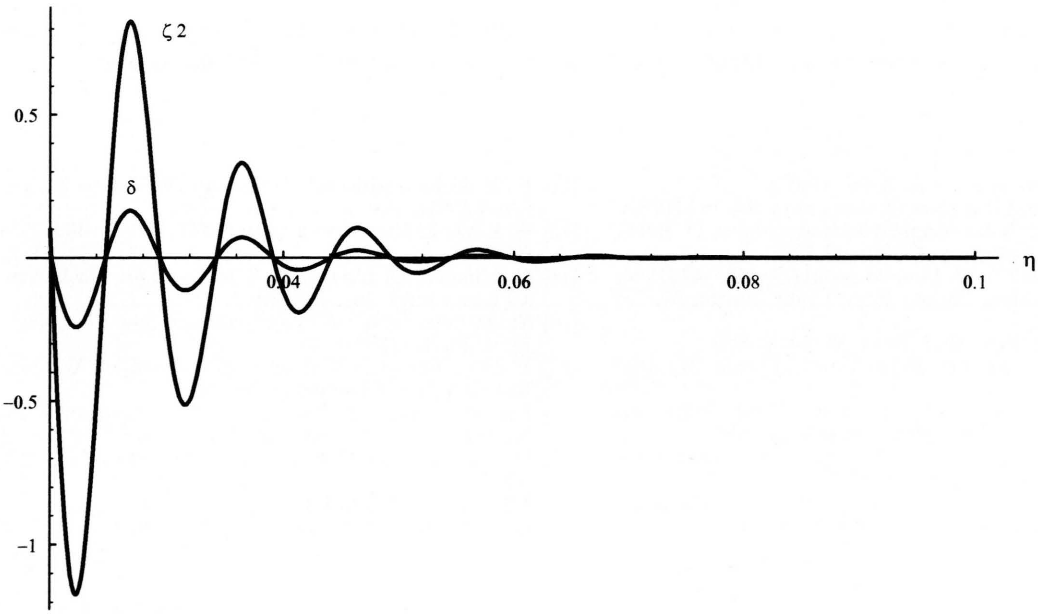

Fig. 12. Plots of $\zeta_2(\eta)$ and $\delta(\eta)$ corresponding to (40) with the same conditions as in Figure 9. Note that the zeros of both functions coincide.

4. Concluding Remarks

Each of the similarity solutions given in Sect. 3 can be used as f in (9) to obtain further similarity solutions. Note that in this way all solutions contained in different subgroups $H^1, \dots H^6$ lead to different new similarity solutions.

With (4) one can calculate both functions α and, say, $\tilde{\alpha}$ for a given similarity solution and the transformed similarity solution (9). It can be shown that $\tilde{\alpha} = e^{-\varepsilon_7} \alpha$ follows. Therefore in the cases $\mathscr{H}^2$, $\mathscr{H}^3$, and $\mathscr{H}^5$ only similarity solutions with $\alpha = \text{const}$ exist. For $\mathscr{H}^4$, the case $B_\varrho = 0$ is a special one because the function α may be chosen in (17). In the case $\mathscr{H}^6$, interesting forms of similarity solutions are given by (22) in terms of ζ_i, $i = 1, 2, 3$, which are plotted for some initial conditions and different values of parameter b in Figs. 1, 2, 3, and 9. These Figures display the different decay of the magnetic field with radial distance at $z = \text{const}$.

The magnetohydrodynamic stability of force-free magnetic fields has been studied by several authors with the aid of the energy principle of Bernstein et al. [16]. The stability of a constant-α force-free field (e.g. Lundquist field) can be achieved by surrounding the plasma by a perfectly conducting rigid wall (cf. Voslamber and Callebaut [17], Krüger [18]). Force-free fields which are describing certain magnetic structures in astrophysical situations are confined hardly by rigid walls. Therefore Chiuderi et al. [13] considered the stability properties of cylindrical force-free fields where α goes smoothly to zero at sufficiently large radial distances. They used (14) and (16) as equilibrium equations and completed the configuration by a large-scale potential field which surrounds the force-free field beyond a certain radius. A general discussion of the stability properties of force-free similarity solutions cannot be given here. But it should be emphasized that some of the similarity solutions discussed in this paper are connected with a nonconstant α which goes smoothly to zero. Corresponding to the numerical similarity solutions, one obtains functions α with different radial dependence within z-planes. Two of them are presented in Figs. 7 and 8. The remarkable oscillatory behavior of α shown in Fig. 8 is similar to the behavior of the same function in a Figure which is to be found in a paper given with respect to practical applications of high-temperature superconductors by Marsh [19].

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